t_n are chosen so that $t_n = 2n\pi/\lambda_0$ (n = any integer), the sine-cosine terms are seen to vanish, and only the first exponential term and constants remain; the radial displacements ϵ_n at time t_n are then given by the particularly simple relation

$$\epsilon_n = \alpha_0(e^{2n\pi\lambda_1/\lambda_0} - 1) \tag{21}$$

and a similar expression results for φ_n. For any conveniently chosen integer n, ϵ_n and φ_n constitute new initial conditions for the next time interval. The accuracy of the final results obtained in this manner is *better than first order* in ϵ and φ; though convergence has not been proved, the assumption seems reasonable that by further subdivision of time intervals Eqs. (14) and (15) yield a sequence of solutions which tends to the exact solution of the problem. For *low thrust* devices that often necessitate integrations over hundreds of orbits, the method just outlined will have computational advantages.

The *inclusion of thrust* in the forementioned formulism involves merely the addition of terms, since the basic equations are linear (cf., the previous footnote). If the Laplace transforms of $\epsilon(t)$ and $\varphi(t)$ are known and if $\bar{F}_1(s)$ and $\bar{F}_2(s)$ are the transforms of thrust-functions $F_1(t)$ and $F_2(t)$, then, to include the thrust, $[A + \bar{F}_1(s)]$ must be inserted in place of A and $[B + \bar{F}_2(s)]$ in place of B, and the inverse transforms then yield the complete response that is the complementary function and a particular integral.

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Comparison of Error Transfer Matrices for Circular Orbits

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IN problems dealing with the motion of orbiting bodies, it is often necessary to investigate the propagation of errors in position and velocity as the body progresses in its orbit. A convenient tool for such studies is the error matrix, which relates position and velocity errors at two arbitrary points of an orbit.

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For circular orbits, there are two matrices in common use. The first, which may be referred to as the Clohessy-Wiltshire[1] matrix, refers to the relative motion between an orbiting body and a reference satellite in circular orbit. The relative vector is expressed in terms of a Cartesian coordinate system centered on the reference satellite and rotating at a constant angular rate in the reference orbital plane to keep one axis always horizontal. The second formulation may be called the Duke[2] matrix because of its use in referenced reports. This also describes the relative motion between an orbiting body and a reference satellite in circular orbit, but a different coordinate system is employed. In this case the quantities employed are the relative differences in distance to the center of the force field, central angle subtended, velocity vector magnitude, and velocity vector angle with respect to the vertical. A recent paper by Wisneski[3] presents a detailed approach to the derivation of the Duke matrix.

Because of the difference in coordinate systems employed, the two formulations appear to be different. Nevertheless, they are in fact exactly equivalent, and each is based upon the same linearizing assumptions and approximations. It is the purpose of this note to show the equivalence of the two formulations. To avoid confusion, the first formulation will be henceforth referred to as Clohessy-Wiltshire, and the second will be called Duke.

Derivation of Transformation Equations

The pertinent geometry is given in Fig. 1. The reference satellite in circular orbit is labeled S; the orbiting body of interest is located at P. A planar situation only is described. Although a third dimension (perpendicular to the orbital plane) and a time perturbation may be added, these have been omitted since the matrix description of each formulation gives identical terms for each quantity.

The satellite is moving counterclockwise at constant translational velocity V, angular velocity ω, and distance R from the center of the force field O. In the Clohessy-Wiltshire system the relative position of the orbiting body is measured with respect to the rotating Cartesian axes labeled x-y. The coordinates are designated Δx, Δy for position, and $\Delta\dot{x}$, $\Delta\dot{y}$ for velocity.

For the Duke system, the reference satellite is described by the parameters R, θ, V, and β. The orbiting body is described by R_p, θ_p, V_p, and β_p. The relative parameters are then

$$\Delta R = R_p - R \qquad \Delta\theta = \theta_p - \theta$$

$$\Delta V = V_p - V \qquad \Delta\beta = \beta_p - \beta$$

Note that β is actually constant at 90° because of the circularity of the reference satellite's orbit.

The relations between the Clohessy-Wiltshire and Duke systems are derived as follows, noting that

$$\Delta R \ll R \qquad \Delta\theta = \text{very small angle}$$

$$\Delta V \ll V \qquad \Delta\beta = \text{very small angle}$$

For horizontal position,

$$\Delta x = -(R + \Delta R)\sin\Delta\theta \cong -R\Delta\theta \tag{1}$$

For vertical position,

$$\Delta y = (R + \Delta R)\cos\Delta\theta - R \cong \Delta R \tag{2}$$

For horizontal velocity,

$$\Delta\dot{x} = -V_p \sin(\beta_p + \Delta\theta) + \omega\Delta y + V$$

Here the $\omega\Delta y$ term arises because of the rotation of the coordinate system.

Note that $\sin(\beta_p + \Delta\theta) = \cos(90° - \beta_p - \Delta\theta) = \cos(-\Delta\beta - \Delta\theta)$, since $\beta = 90°$. Since $\Delta\beta$ and $\Delta\theta$ are very small,

$$\Delta\dot{x} \cong -(V_p - V) + \omega\Delta y$$

or

$$\Delta\dot{x} \cong -\Delta V + \omega\Delta y$$

Substituting relation (2),

$$\Delta\dot{x} = -\Delta V + \omega\Delta R \qquad (3)$$

For vertical velocity,

$$\Delta\dot{y} = V_p \cos(\beta_p + \Delta\theta) - \omega\Delta x$$

But

$$\cos(\beta_p + \Delta\theta) = -\sin(\Delta\beta + \Delta\theta) \cong -\Delta\beta - \Delta\theta$$

since $\beta = 90°$. Also, $\Delta x \cong -R\Delta\theta$ from relation (1), and $V_p = V + \Delta V$. Hence,

$$\Delta\dot{y} \cong -(V + \Delta V)(\Delta\beta + \Delta\theta) + \omega R\Delta\theta$$

Noting that $V = \omega R$ and neglecting second-order differences, this becomes

$$\Delta\dot{y} \cong -V\Delta\beta \qquad (4)$$

Collecting the results of the preceding analysis, the conversions from the Duke to the Clohessy-Wiltshire systems are

$$\Delta x \cong -R\Delta\theta \qquad \Delta y \cong \Delta R$$

$$\Delta\dot{x} \cong -\Delta V + \omega\Delta R \qquad \Delta\dot{y} \cong -V\Delta\beta$$

By algebraic manipulation of relations (1–4), the following inverse results are obtained:

$$\Delta R \cong \Delta y \qquad (5)$$

$$\Delta\theta \cong -\Delta x/R \qquad (6)$$

$$\Delta V \cong -\Delta\dot{x} + \omega\Delta y \qquad (7)$$

$$\Delta\beta \cong -\Delta\dot{y}/V \qquad (8)$$

The two-dimensional forms of the Clohessy-Wiltshire and the Duke matrices are as follows:

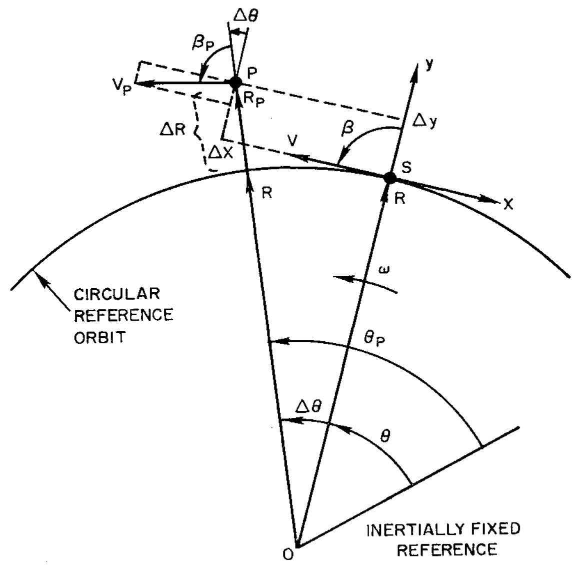

Fig. 1 Coordinate systems.

hessy-Wiltshire system). As an illustration, the equation for $\Delta\dot{x}_t$ is formed below.

Rewriting Eq. (3),

$$\Delta\dot{x}_t = -\Delta V_t + \omega\Delta R_t$$

Now substitute for ΔV_t and ΔR_t, using the Duke transfer relations:

$$\Delta\dot{x}_t = -[\omega(\cos\omega t - 1)\Delta R_i + (2\cos\omega t - 1)\Delta V_i + (V\sin\omega t)\Delta\beta_i] + \omega[(2 - \cos\omega t)\Delta R_i + (2/\omega) \times (1 - \cos\omega t)\Delta V_i - (R\sin\omega t)\Delta\beta_i]$$

After collecting terms and noting that $V = \omega R$, this becomes

$$\Delta\dot{x}_t = \omega(3 - 2\cos\omega t)\Delta R_i + (3 - 4\cos\omega t)\Delta V_i - (2V\sin\omega t)\Delta\beta_i$$

Equations (5, 7, and 8) are now used to substitute for ΔR_i, ΔV_i, and $\Delta\beta_i$:

Clohessy-Wiltshire Equations

$$\begin{bmatrix} \Delta x_t \\ \Delta y_t \\ \Delta\dot{x}_t \\ \Delta\dot{y}_t \end{bmatrix} = \begin{bmatrix} 1 & 6(\omega t - \sin\omega t) & \frac{4}{\omega}\sin\omega t - 3t & \frac{2}{\omega}(1 - \cos\omega t) \\ 0 & 4 - 3\cos\omega t & -\frac{2}{\omega}(1 - \cos\omega t) & \frac{1}{\omega}\sin\omega t \\ 0 & 6\omega(1 - \cos\omega t) & 4\cos\omega t - 3 & 2\sin\omega t \\ 0 & 3\omega\sin\omega t & -2\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} \Delta x_i \\ \Delta y_i \\ \Delta\dot{x}_i \\ \Delta\dot{y}_i \end{bmatrix}$$

Duke Equations

$$\begin{bmatrix} \Delta R_t \\ \Delta\theta_t \\ \Delta V_t \\ \Delta\beta_t \end{bmatrix} = \begin{bmatrix} 2 - \cos\omega t & 0 & \frac{2}{\omega}(1 - \cos\omega t) & -R\sin\omega t \\ \frac{1}{R}(2\sin\omega t - 3\omega t) & 1 & \frac{1}{V}(4\sin\omega t - 3\omega t) & 2(1 - \cos\omega t) \\ \omega(\cos\omega t - 1) & & 2\cos\omega t - 1 & V\sin\omega t \\ -\frac{1}{R}\sin\omega t & 0 & -\frac{2}{V}\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} \Delta R_i \\ \Delta\theta_i \\ \Delta V_i \\ \Delta\beta_i \end{bmatrix}$$

where subscript t indicates values at time t after initial conditions, and subscript i denotes initial conditions.

Illustrative Transformation from Duke Matrix to Clohessy-Wiltshire Matrix

The Duke transfer equations express relative position and velocity at an arbitrary time t as a function of the initial values (i.e., relative position and velocity at the initial time). Their matrix is given in terms of a ΔR, $\Delta\theta$, ΔV, $\Delta\beta$ system. Using the transformations just derived, their matrix may be expressed in terms of a Δx, Δy, $\Delta\dot{x}$, $\Delta\dot{y}$ system (i.e., the Clohessy-Wiltshire system).

$$\Delta\dot{x}_t = \omega(3 - 2\cos\omega t)\Delta y_i + (3 - 4\cos\omega t) \times (-\Delta\dot{x}_i - \omega\Delta y_i) - (2V\sin\omega t)[-(\Delta\dot{y}_i/V)]$$

After expanding and collecting terms, the following result is obtained:

$$\Delta\dot{x}_t = 6\omega(1 - \cos\omega t)\Delta y_i + (4\cos\omega t - 3)\Delta\dot{x}_i + (2\sin\omega t)\Delta\dot{y}_i \qquad (9)$$

Equation (9) is the desired Clohessy-Wiltshire transformation.

Summary

The foregoing analysis shows that the Clohessy-Wiltshire and Duke matrices are equivalent. They stem, in fact, from the same linearizing approximations. The principal assumptions are 1) the distance between orbiting body and reference satellite is very small compared to the distance to the center of the force field; and 2) the variation of the gravity field is linear in the vertical direction over the region of interest.

Examination of the transformation equations reveals an interesting lack of symmetry, i.e., the horizontal velocity transformation is cross-coupled with the vertical position. Yet no coupling term appears for the vertical velocity. The coupling is caused by the coordinate system rotation. Since the Clohessy-Wiltshire system rotates with respect to inertial space, the transformation between it and a fixed inertial system contains symmetrical cross-coupling terms for each velocity axis. The lack of symmetry in the transformation equations under discussion may be interpreted to mean that the Duke system appears to be rotating when vertical velocities are considered but appears to be fixed for horizontal velocities. Hence the correct translation of velocities for this system to those in an inertial coordinate system requires considerable care.

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Particle Damping of a Plane Acoustic Wave in Solid Propellant Combustion Gases

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Nomenclature

a = particle radius, cm
c = velocity of sound in the gas, cm/sec
c_v = specific heat of the gas at constant volume, cal/g-°K
E = energy flux in the plane wave, erg/cm²-sec
f = acoustic frequency, cps
H'' = defined by Eq. (2), dimensionless
k = ω/c, cm^{-1}
L = decay length for all three mechanisms combined, cm
L_b = decay length for bulk damping of the pure gas phase, cm
L_p = decay length for particle damping, cm
L_w = decay length for pure gas phase wall damping, cm
N = particle number density in combustion products, cm^{-3}
P_0 = equilibrium pressure, dynes/cm²
R = tube radius, cm
V = volume of condensed products per unit volume of total combustion products
x = distance, cm
β = $(\omega/2\nu)^{1/2}$, cm^{-1}
γ = ratio of specific heats
λ = gas phase thermal conductivity, cal/sec-cm-°K
μ = dynamic viscosity, poise
ν = kinematic viscosity, stokes
ρ_0 = gas phase density, g/cm³
ρ_1 = solid particle density, g/cm³
σ = acoustic damping constant, cm^{-1}
ω = circular frequency, rad/sec

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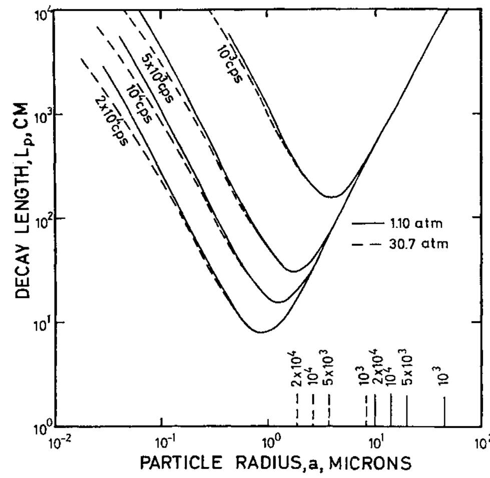

Fig. 1 Decay length for particle damping as a function of particle radius. For the indicated frequencies, the short vertical lines give the largest particle radii for which the calculation is valid. The calculation was restricted to $\beta a <$ 10–1.

It is a matter of general knowledge that acoustic combustion instability in solid propellant rocket motors can often be cured by adding certain substances to the solid propellant itself. These substances may or may not participate in the chemical reactions of the combustion process, but they always produce solid or liquid particles in the combustion gases. Additives such as Al and Mg powders, which participate in the combustion reactions, appear to be more effective in suppressing acoustic instability than do inert additives.

The mechanism by which these additives suppress instability is not known. They may act by decreasing the effectiveness of the acoustic amplifiers in the motor or by increasing the acoustic losses. It is the purpose of this note to present calculations that investigate the latter possibility. The viscous damping that is caused by the presence of solid or liquid particles in the combustion gases is calculated and compared with that of the pure gas phase without particles. The particular case of a propellant that contains 10% aluminium by weight is considered. A plane acoustic wave is assumed, and the effect of particle size on the decay length of the wave is calculated for frequencies from 10^3 to 2×10^4 cps and for pressures of 1 and 28 atm. The results are compared with the decay length for thermal and viscous damping in the bulk of the pure gas phase, and with the decay length for thermal and viscous wall damping of the pure gas phase when the wave is traveling axially along a cylindrical tube of 5-cm radius.

Previous work concerning the effect of particle damping on a particular mode of oscillation in a solid propellant rocket motor may be found in Ref. 1.

Analysis

Reference 2 gives the results of a theory of the energy absorbed by viscous damping from a plane acoustic wave as it passes over a spherical particle that is free to move.

For the case where the plane wave is propagated through a medium that contains many such particles, all of radius a, and where the particles are not so closely spaced that they interact with one another, the rate of dissipation of acoustic